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Bounded generalized Dedekind prime rings

by
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Dedicated to the memory of Nicolae Popescu (1937-2010)
on the occasion of his 75th anniversary

Abstract

We give several characterizations of bounded generalized Dedekind prime rings in terms of invertible prime ideals and provide examples of PI generalized Dedekind prime rings in which every maximal ideal is localizable.

Key Words: Generalized Dedekind prime ring, Invertible ideal, Localizable, Prime v-ideal.
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1 Introduction

In [2], one of the authors has introduced a new class of rings, called generalized Dedekind prime rings (for short, $G-$Dedekind prime rings) and studied the structure of them (see also [3]).

The aim of this paper is to characterize bounded $G-$Dedekind prime rings in terms of invertible prime ideals (without the assumption of maximal orders), which are, in a sense, a generalization of [3, Theorem 2.6].

Let R be a prime Goldie ring with its quotient ring Q. For any (fractional) right $R-$ideal I and left $R-$ideal J, let

$$(R:I)_l = \{q \in Q | qI \subseteq R\} \text{ and } (R:J)_r = \{q \in Q | Jq \subseteq R\}$$

which is a left (right) $R-$ideal, respectively and

$$I_v = (R:(R:I)_l)_r \text{ and } {}_vJ = (R:(R:J)_r)_l$$

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which is a right (left) $R-$ideal containing $I(J)$. $I(J)$ is called a *right (left) $v-$ideal* if $I_v = I$ ($_vJ = J$). In case I is a two-sided $R-$ideal it is said to be a $v-$ideal if $I_v = I =_v I$. An $R-$ideal A is said to be a *$v-$invertible* if $_v((R : A)_l A) = R = (A(R : A)_r)_v$. Note if A is $v-$invertible, then $(R : A)_l = A^{-1} = (R : A)_r$ and $O_l(A) = R = O_r(A)$, where $A^{-1} = \{q \in Q | AqA \subseteq A\}$, $O_l(A) = \{q \in Q | qA \subseteq A\}$ and $O_r(A) = \{q \in Q | Aq \subseteq A\}$. Of course, a $v-$invertible ideal is invertible.
For any unexplained terminology we refer to [9].

2 Characterizations of bounded generalized Dedekind prime rings

Throughout this paper, R is a prime Goldie ring with its quotient ring Q. We start with the following two lemmas which are more or less known.

Lemma 2.1. *Let R be a prime Goldie ring and A be an $R-$ideal.*

(1) For any right $R-$ideal I, $(IA)_v = (IA_v)_v$ and if A is invertible, then $(I_vA)_v = (IA)_v$.

(2) If A is invertible and B is an $R-$ideal with $B = B_v$, then $(BA)_v = BA$.

Proof: (1) See [7, Lemma 1.1].
(2) Suppose that $(BA)_v \supset BA$. Then $B = BAA^{-1} \subset (BA)_v A^{-1} \subseteq ((BA)_v A^{-1})_v =$
$= ((BA)A^{-1})_v = B_v = B$, a contradiction. Hence $(BA)_v = BA$ follows. □

Lemma 2.2. *Let C be a reguler Ore set of a Noetherian prime ring R and $S = R_C$. Let I be a right $R-$ideal and J be a left $R-$ideal. Then*

(i) $(S : IS)_l = S(R : I)_l$ and $(S : SJ)_r = (R : J)_v S$.

(ii) $(IS)_v = I_v S$ and $_v(SJ) = S_v J$.

Proof: (i) It is clear that $(S : IS)_l \supseteq S(R : I)_l$. Let $x \in (S : IS)_l$. Then $xI \subseteq xIS \subseteq S$. Since I is a finitely generated as a right ideal, there is a $c \in C$ with $cxI \subseteq R$, that is, $cx \in (R : I)_l$ and so $x \in c^{-1}(R : I)_l \subseteq S(R : I)_l$. Hence $(S : IS)_l = S(R : I)_l$. Similarly $(S : SJ)_r = (R : J)_r S$ follows. The second statements (ii) follows from (i). □

A Noetherian prime ring is called a *generalized Dedekind prime ring* if it is a maximal order and each $v-$ideal is invertible ([2]). Without the assumption on maximal orders we have the following :

Proposition 2.3. *For a Noetherian prime ring R the following conditions are equivalent :*

(1) R is a $G-$Dedekind prime ring.

(2) Every prime ideal P of R with $P = P_v$ (or $P =_v P$) is invertible.

(3) Every ideal A of R with $A = A_v$ (or $A =_v A$) is invertible.

Proof: (1) $\Rightarrow$ (3) : Let A be a (fractional) $R-$ideal. It suffices to prove that ${}_vA = A_v$. Since R is a maximal order, $(R:A)_l = A^{-1} = (R:A)_r$. Thus $A_v = (R:(R:A)_l)_r = (R:A^{-1})_r = A^{-1-1}$ and similarly ${}_vA = A^{-1-1}$. Hence ${}_vA = A_v$ follows.
(3) $\Rightarrow$ (2) : This is a special case.
(3) $\Rightarrow$ (1) : For each ideal A, we have $R \subseteq O_l(A) \subseteq O_l(A_v)$ which is equal to R, because $A_v = (A_v)_v$ and is invertible by the assumption. Hence $O_l(A) = R$ and similarly $O_r(A) = R$, that is, R is a maximal order and so it is a $G-$Dedekind prime ring.
(2) $\Rightarrow$ (3) : Set $\mathfrak{A} = \{A : \text{ ideal of } R | A = A_v\}$. If A is a maximal element in $\mathfrak{A}$, then it is a prime ideal by Lemma 2.1 (1) and so it is invertible. Assume that there exists an $A \in \mathfrak{A}$ which is not invertible. We may assume that A is a maximal one for this property. There is an invertible prime ideal P with $P \supset A$ and $R = PP^{-1} \supset AP^{-1} \supseteq A$. If $AP^{-1} = A$, then $A = AP$. Consider the localization R_P of R at P, which is a local principal ideal ring such that $\{p^n R_P | n = 1, 2, ...\}$ is the set of all proper ideals of R_P, where $PR_P = pR_P$ ([5]). Since AR_P is an ideal of R_P ([10, (2.1.16)]), we have, for some $n \geq 1$, $p^n R_P = AR_P = APR_P = p^{n+1}R_P$ which entails $R_P = pR_P$, a contradiction. Thus $AP^{-1} \supset A$ and $(AP^{-1})_v = AP^{-1}$ by Lemma 1.1. Hence, by the choice of A, AP^{-1} is invertible and so is A, which is a contradiction. This completes the proof. □

From Proposition 2.3, we have the following corollary :

Corollary 2.4. *Let R be a Noetherian prime ring. Suppose that each prime ideal contains an invertible prime ideal. Then R is a $G-$Dedekind prime ring.*

Proof: Let P be a prime ideal with $P = P_v$. Then there exists an invertible prime ideal P_1 with $P \supseteq P_1$. It follows that $P_1(R:P)_l \subseteq P_1(R:P_1)_l = P_1(R:P_1)_r = R$ since $(R:P_1)_l = (R:P_1)_r$. Note that $P_1(R:P)_l P \subseteq P_1$. If $P \supset P_1$, then $P_1(R:P)_l \subseteq P_1$ and $(R:P)_l \subseteq O_r(P_1) = R$. Thus $P_v = R$, a contradiction, that is $P = P_1$ follows. Similarly if $P =_v P$, then it is invertible. Hence R is a $G-$Dedekind prime ring by Proposition 2.3. □

If R is a bounded $G-$Dedekind prime ring, then the converse of Corollary 2.4 is also true : Let P be a non-zero prime ideal of a bounded $G-$Dedekind prime ring and let c be a regular element in P. Then cR contains a non-zero ideal A with $A = A_v$. By Proposition 2.3 and [2, Theorem 3.1], P contains an invertible prime ideal.
However the converse of Corollary 2.4 is not necessarily held as we will give a counter example in the end of the paper.

A Noetherian prime ring is called a *unique factorization ring* (a Noetherian UFR for short) in the sense of [6] if every non-zero prime ideal contains a nonzero principal prime ideal. In [1], they defined another UFRs by using $v-$ideals : R is called a *UFR* if every prime ideal P of R with $P = P_v$ (or $P =_v P$) is principal. It follows that a UFR in the sense of [6] implies a UFR in the sense of [1] and that the converse is not necessarily held (see [1]). However, every bounded UFR in the sense of [1] implies a UFR in the sense of [6]. This is proved in the similar way as in case of bounded $G-$Dedekind prime rings.

In [3] she gave new characterizations of $G-$Dedekind prime rings under the PI condition. The following theorem is, in a sense, a generalization of [3, Theorem 2.6] to the case of bounded $G-$Dedekind prime rings.

Theorem 2.5. *Let R be a bounded prime Notherian ring. Then the following conditions are equivalent :*

(1) R is a $G-$Dedekind prime ring.

(2) For every regular element c of R, cR and Rc contain a finite product of invertible prime ideals, respectively.

(3) Every prime ideal of R contains an invertible prime ideal.
Moreover, if every maximal ideal of R is localizable then the conditions $(1)-(3)$ are equivalent to :

(4) For every maximal ideal M, the localized ring R_M is a UFR in the sense of [6].

Proof: $(1) \Rightarrow (2)$: Let c be a regular element of R. Then cR contains a non-zero ideal A. We may assume that $A = A_v$ and it is a finite product of invertible prime ideals by Proposition 2.3 and [2, Theorem 3.1].
$(2) \Rightarrow (3)$: Let P be a prime ideal and c be a reguler element in P. Then there are a finite invertible prime ideals $P_1, ..., P_n$ such that $P \supseteq cR \supseteq P_1...P_n$ and so $P \supseteq P_i$ for some i.
$(3) \Rightarrow (1)$: This follows from Corollary 2.4.
Now suppose that every maximal ideal of R is localizable.
$(3) \Rightarrow (4)$: Let M be a maximal ideal of R and let P' be a prime ideal of R_M. Then $P = P' \cap R$ is a prime ideal of R by [10, (2.1.16)]. There exists an invertible prime ideal P_1 with $P \supseteq P_1$. Then $P_1 R_M$ is a prime ideal of R_M which is invertible such that $P' \supseteq P_1 R_M$. By [4, Lemma 3.4], $P_1 R_M$ is principal. Hence R_M is a UFR in the sense of [6].
$(4) \Rightarrow (1)$: Let P be a prime ideal of R with $P = P_v$. Suppose $(R : P)_l P \neq R$. Then there exists a maximal ideal M of R such that $M \supseteq (R : P)_l P \supseteq P$. Put $P' = PR_M$, a prime ideal of R_M with $P' \cap R = P$. By Lemma 2.2, $P'_v = P_v R_M = PR_M = P' \supseteq P$. Thus P' is principal, say, $P' = pR_M$ for some $p \in P$ and $R_M \supseteq p^{-1}P$. Since $p^{-1}P$ is a finitely generated right $R-$ideal, there is a $c \in C(M)$ with $cp^{-1}P \subseteq R$, that is, $cp^{-1} \in (R : P)_l$ and $c \in (R : P)_l P \subseteq M$, a contradiction. Hence $(R : P)_l P = R$. Furthermore $R_M {}_v P =_v (R_M P) =_v (PR_M) = PR_M$ since PR_M is principal, which implies ${}_v P \subseteq PR_M \cap R = P$, that is ${}_v P = P$. Thus, by left version of the discussion above, $P(R : P)_r = R$. Hence P is invertible and R is a $G-$Dedekind prime ring by Proposition 2.3. □

In [3] she did not give an example of a $G-$Dedekind prime ring in which every maximal ideal is localizable. We may give such examples.

Let D be a commutative semi-local Dedekind domain with maximal ideals $\mathfrak{m}_i$ $(1 \leq i \leq n)$ and σ be an automorphism of D such that $\sigma^2 = 1$ and $\sigma(\mathfrak{m}_i) = \mathfrak{m}_i$ for each i. Put $R = D[[x; \sigma]]$, the skew formal power series ring over R in an indeterminate x. Then we have the following properties :

(1) R is a Noetherian maximal order ([8, §1]).

(2) gl.dim $R = 2$ ([10, (7.5.3)]).

(3) R is a $G-$Dedekind prime ring ([2]).

(4) $M_i = \mathfrak{m}_i + xR (1 \leq i \leq n)$ are only maximal ideals of R.

Proof: Since $\dfrac{R}{M_i} \cong \dfrac{D}{\mathfrak{m}_i}$, it follows that M_i are maximal ideals of R. Suppose N is a maximal ideal of R which is different from M_i. Then $N + M_i = R$ for each i and so $R = (N + M_1)(N + M_2)...(N + M_n) \subseteq N + M_1M_2...M_n \subseteq R$, that is, $R = N + M_1M_2...M_n$. Write $1 = a + b$, where $a = a_0 + a_1x + a_2x^2 + ... \in N$ and $b = b_0 + b_1x + b_2x^2 + ... \in M_1...M_n$, where $b_0 \in \mathfrak{m}_1...\mathfrak{m}_n \subseteq J(D)$, the Jacobson radical of D. Thus we have $1 = a_0 + b_0$ and $a_0 = 1 - b_0 \in 1 + J(D)$, that is, a_0 is a unit in D. This means a is a unit in R which is a contradiction.
In what follows, put $M = \mathfrak{m} + xR$ and $\mathfrak{m}$ is a maximal ideal of D. Since $\sigma^2 = 1$, the center $\mathbb{Z}(R)$ of R is $D_\sigma[[x^2]]$, where $D_\sigma = \{a \in D | \sigma(a) = a\}$. Furthermore $C(M) = \{a = a_0 + a_1x + a_2x^2 + ... \in R | a_0 \in D - \mathfrak{m}\}$. □

Lemma 2.6. *Under the same notation above, let $a = a_0 + a_1x + a_2x^2 + ... \in C(M)$. Then there are $b = b_0 + b_1x + b_2x^2 + ... \in R$ and $c = c_0 + c_2x^2 + ... \in C(M) \cap \mathbb{Z}(R)$ such that $ab = c$*

Proof: We define b_i and c_i in the following way :
Define $b_0 = \sigma(a_0)$ and $c_0 = a_0\sigma(a_0) \in D_\sigma - \mathfrak{m}_\sigma$, because $\sigma(\mathfrak{m}) = \mathfrak{m}$, where $\mathfrak{m}_\sigma = D_\sigma \cap \mathfrak{m}$. For a natural number n we define $b_n = \sigma(a_n)$ if n is even and $b_n = -a_n$ if n is odd. Furthermore, we define $c_n = \sum_{k=0}^{n} a_k\sigma^k(b_{n-k})$.
In case $n = 2j$. We have

$$c_{2j} = \sum_{k=0}^{j-1}(a_k\sigma^k(b_{2j-k}) + a_{2j-k}\sigma^{2j-k}(b_k)) + a_j\sigma^j(b_j). \tag{1}$$

If k is even then $a_k\sigma^k(b_{2j-k}) + a_{2j-k}\sigma^{2j-k}(b_k) = a_k\sigma(a_{2j-k}) + a_{2j-k}\sigma(a_k) \in D_\sigma$.
If k is odd, then $a_k\sigma^k(b_{2j-k}) + a_{2j-k}\sigma^{2j-k}(b_k) = a_k\sigma(-a_{2j-k}) + a_{2j-k}\sigma(-a_k) \in D_\sigma$.
Finally $a_j\sigma^j(b_j) = a_j\sigma(a_j) \in D_\sigma$ if j is even and $a_j\sigma^j(b_j) = a_j\sigma(-a_j) \in D_\sigma$ if j is odd. Hence $c_{2j} \in D_\sigma$ follows.
In case $n = 2j + 1$. We have

$$c_{2j+1} = \sum_{k=0}^{j}(a_k\sigma^k(b_{2j+1-k}) + a_{2j+1-k}\sigma^{2j+1-k}(b_k)). \tag{2}$$

If k is even, then $a_k\sigma^k(b_{2j+1-k}) + a_{2j+1-k}\sigma^{2j+1-k}(b_k) = a_k(-a_{2j+1-k}) + a_{2j+1-k}a_k = 0$.
If k is odd, then $a_k\sigma^k(b_{2j+1-k}) + a_{2j+1-k}\sigma^{2j+1-k}(b_k) = a_ka_{2j+1-k} + a_{2j+1-k}(-a_k) = 0$.
Hence $c_{2j+1} = 0$ follows. Hence $ab = c$ and $c \in C(M) \cap \mathbb{Z}(R)$. □

Lemma 2.7. *Under the same notation as in Lemma 2.6, M is localizable and R_M is a UFR in the sense of [6].*

Proof: $C = C(M) \cap \mathbb{Z}(R) = \{c = c_0 + c_2x^2 + ... \in D_\sigma[[x^2]] | c_0 \in D_\sigma - \mathfrak{m}_\sigma\}$ is closed under the multiplication and so it is an Ore set of R. For each $a \in R$ and $d \in C(M)$, there are $b \in R$ and $c \in C$ with $db = c$. Hence $ac = ca = dba$, that is, $C(M)$ is an Ore set of R. It is clear that $R_M = R_C$. It is also obvious that R is a prime PI ring, because R is a finitely generated $D_\sigma[[x^2]]-$module. Since R is a $G-$Dedekind prime ring, it follows that R_M is a UFR in the sense of [6] by [3, theorem 2.6] □

Summarizing the observation above, we have the following example :

Example 1. *Let D be a commutative semi-local Dedekind domain and σ be an automorphism of D such that $\sigma^2 = 1$ and $\sigma(\mathfrak{m}) = \mathfrak{m}$ for each maximal ideal $\mathfrak{m}$. Then $R = D[[x;\sigma]]$ is a $G-$Dedekind prime PI-ring in which each maximal M of R is localizable and R_M is a UFR in the sense of [6].*

We can easily find a plenty of commutative semi-local Dedekind domains satisfying the conditions in Example 1 in number theory and polynomial rings. We only give a simple example in polynomial rings : Let $K[x,y]$ be a polynomial ring over a field K of characteristic$\neq 2$ in indeterminates x, y and σ be an automorphism of $K[x,y]$ defined by $\sigma(x) = -x$ and $\sigma(y) = -y$ so that $\sigma^2 = 1$. Put $D = K[x,y]_{(x)} \cap K[x,y]_{(y)}$, which is a semi-local Dedekind domain with only two maximal ideals $\mathfrak{m}_1 = xD$, and $\mathfrak{m}_2 = yD$ and $\sigma(\mathfrak{m}_i) = \mathfrak{m}_i$ $(i = 1, 2)$.

We end the paper with an example mentioned of after Corollary 2.4 ([6, Example 5.2]) : Let $R = K[t,y]$ be the polynomial ring over a field K of characteristic zero in indeterminates t, y and $\delta = 2y\frac{\partial}{\partial t} + (y^2 + t)\frac{\partial}{\partial y}$, a derivation of R. Then the only height-1 prime ideals are $\mathfrak{p}_1[x;\delta]$ and $\mathfrak{p}_2[x;\delta]$, where $\mathfrak{p}_1 = (y^2 + t + 1)R$ and $\mathfrak{p}_2 = tR + yR$ which are only non-zero δ-prime ideals of R. Since $(\mathfrak{p}_2[x;\delta]_v) = (\mathfrak{p}_2)_v[x;\delta] = R[x;\delta]$ (see the proof of [11, Lemma 3]), it follows that $\mathfrak{p}_2[x;\delta]$ is not invertible. Hence $R[x;\delta]$ is not satisfied the assumption in Corollary 2.4. By [1, Proposition 3.1], $R[x;\delta]$ is a UFR in the sense of [1] and so it is a $G-$Dedekind prime ring.

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