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A Study on Anosov Diffeomorphisms on Real Bott Manifolds and Acyclic Digraphs

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Abstract

We provide a characterization of real Bott manifolds admitting Anosov diffeomorphisms in terms of acyclic digraphs.

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1 Introduction

By the definition of [13], [14] *a real Bott manifold* is the total space B_n of a sequence of $\mathbb{R}P^1$-bundles starting with a point:

$$B_n \to B_{n-1} \to \cdots \to B_2 \to B_1 \to \{\text{a point}\}.$$

Each $\mathbb{R}P^1$-bundle $B_i \to B_{i-1}$ is the projectivization of the Whitney sum of a real line bundle L_i and the trivial line bundle over B_{i-1}. Futhermore in [13], [14] it was explained that from the viewpoint of group actions, an n-dimensional real Bott manifold (n-RBM) is the quotient of the n-dimensional torus $T^n = S^1 \times \cdots \times S^1$ by the product $(\mathbb{Z}_2)^n$ of cyclic group of order 2. The free action of $(\mathbb{Z}_2)^n$ on T^n can be expressed by an n-th upper triangular matrix A whose diagonal entries are 0 and the other entries are either 1 or 0. The orbit space $M(A) = T^n/(\mathbb{Z}_2)^n$ is the n-dimensional the real Bott manifold and we call A a *Bott matrix* of size n. The real Bott manifold $M(A)$ also provides an example of a flat Riemannian manifold.

Let $f\colon M \to M$ be a diffeomorphism of a flat manifold M. We say f is an affine diffeomorphism of M if the lifting of f to universal covering $\mathbb{R}^n$ of M belongs to $Aff(\mathbb{R}^n) = \mathbb{R}^n \rtimes GL(n, \mathbb{R})$ the group of affine transformation of $\mathbb{R}^n$. Such a lifting then automatically belongs to the normalizer $N_{Aff(\mathbb{R}^n)}(\pi)$ of the fundamental group $\pi = \pi_1(M)$ of M. An affine diffeomorphism f of M is hyperbolic if and only if f lifts to an affine transformation $(d, D) \in Aff(\mathbb{R}^n)$ with D hyperbolic (i.e. having no eigen-values of absolute value 1). As to the denition of Anosov diffeomorphism, you defined for any Riemannian metric *df* decomposes the tangent space into expansion-part and contraction-part. It had better mention that if *df* satises the Anosov property, then *df* satisfies for any Riemannian metric. So we take the euclidean metric (flat metric) on a flat manifold M to show that f is an Anosov diffeomorphism.

Note that each hyperbolic diffeomorphism of a flat Riemannian manifold M defines an Anosov diffeomorphism on M. Conversely, K. Dekimpe et al. [6] proved the following lemma.

Lemma 1.1 ([6])**.** *If $f\colon M \to M$ is an Anosov diffeomorphism of a flat Riemannian manifold M, then f is homotopic to a hyperbolic diffeomorphism $g\colon M \to M$.*

Recall that if π is any crystallographic group, then π satisfies an exact sequence

$$0 \to \Lambda \to \pi \to \Phi \to 1$$

where $\Lambda = \pi \cap \mathbb{R}^n$ is a lattice of rank n, and $\Phi = p_r(\pi)$ is a finite group. Here $p_r\colon \mathrm{E}(n) \to \mathrm{O}(n)$ is a homomorphism defined by $p_r(d, D) = D$, $(d, D) \in \mathrm{E}(n)$. We call Φ, the *holonomy group* of π (see [7]).

Flat Riemannian manifolds supporting Anosov diffeomorphisms can be characterized by the following theorem:

Theorem 1.2 ([5])**.** *An n-dimensional flat manifold M with holonomy group F and associated holonomy representation $T\colon F \to GL(n, \mathbb{Z})$ admits an Anosov diffeomorphism if and only if each $\mathbb{Q}$-irreducible component of T of multiplicity one is reducible over $\mathbb{R}$.*

Here, "multiplicity" means the number of times a component appears in the decomposition of T.

In [13] it was introduced three operations, which are called moves, to a Bott matrix A under which the diffeomorphism class of $M(A)$ does not change. By an iteration of the moves, $M(A)$ is diffeomorphic to $T^k \times_{(\mathbb{Z}_2)^s} M(B)$. That is, there is a k-torus action on $M(A)$ whose quotient space is an $(n-k)$-dimensional real Bott orbifold $M(B)/(\mathbb{Z}_2)^s$ by some $(\mathbb{Z}_2)^s$-action $(1 \le s \le k)$.

Here $M(B)$ is an $(n-k)$-dimensional real Bott manifold corresponding to a Bott matrix B of size $(n-k)$. In addition, each of such $M(A)$ has the first Betti number $b_1(M(A)) = k$, $1 \leq k \leq n$. This result also implies that the Bott matrix A reduces to

$$\left(\begin{array}{c|c|c} O_{k-s} & 0 & 0 \\ \hline 0 & O_s & * \\ \hline 0 & 0 & B \end{array}\right). \tag{1}$$

It follows that

$$M(A) = T^n/(\mathbb{Z}_2)^n = \mathbb{R}^n/\pi(A).$$

Here $\pi(A)$ is the fundamental group of $M(A)$.

On the other hand, an n-RBM is a *small cover* over an n-cube (see [10]). (See definition of a *small cover* in [8].) Choi [9] showed that small covers over cubes are strongly related to *acyclic digraphs*. If D is an acyclic digraph then its adjacency matrix A_D with respect to its acyclic ordering (i.e., if (v_i, v_j) is an arc in D then $i < j$) is an upper triangular matrix with zero diagonals. Choi and Oum [10] introduced two operations, the so-called *local complementation* and *slide*, on acyclic digraphs and proved that two acyclic digraphs D and H are *Bott equivalent* (i.e., one has an isomorphic digraph that is obtained from the other by successively applying the two operations) if and only if the corresponding real Bott manifolds $M(A_D)$ and $M(A_H)$ are diffeomorphic.

In this paper we provide the characterization of real Bott manifolds admitting Anosov diffeomorphisms in terms of acyclic digraphs.

2 Bott equivalence of acyclic digraphs

In this section we review the terminology in graph theory, and recall two operations on acyclic digraphs and some results in [10].

A *directed graph* (or just *digraph*) $D = (V, E)$ consists of a non-empty finite set $V := V(D)$ of elements called *vertices* and a finite set $E := E(D)$ of ordered pairs $e = (u, v)$ of distinct vertices called *arcs*. We call $V(D)$ the *vertex set* and $E(D)$ the *arc set* of D. The *order* of D (denoted by $|D|$) is the number of vertices in D. An ordering $v_1, v_2, \ldots, v_n$ of vertices of a digraph D is an *acyclic ordering* if $i < j$ for each arc (v_i, v_j) in D. A digraph is *acyclic* if it admits an acyclic ordering.

If (u, v) is an arc of D, we say that a vertex u is *adjacent* to a vertex v in D, v is called an *out-neighbor* of u and u is called an *in-neighbor* of v. For a vertex $v \in D$, we use the following notation:

$$N_D^+(v) = \{u \in V\backslash\{v\} | (v, u) \in E\},\ N_D^-(v) = \{u \in V\backslash\{v\} | (u, v) \in E\}.$$

$deg_D^+(v) = |N_D^+(v)|$ is the number of out-neighbors of v, and $deg_D^-(v) = |N_D^-(v)|$ is the number of in-neighbors of v.

For two graphs $D = (V, E)$ and $D' = (V', E')$, a bijection $f\colon V \to V'$ is called an *isomorphism* when $(u, v) \in E$ if and only if $(f(u), f(v)) \in E'$. Two digraphs are *isomorphic* if there is an isomorphism.

For a graph $D = (V, E)$ with a fixed ordering $\{v_1, v_2, \dots, v_n\}$ of V, the *adjacency matrix* of D is an $n \times n$ matrix $A_D = (a_{ij})_{i,j \in 1,2,\dots,n}$ such that

$$a_{ij} = \begin{cases} 1 & \text{if } (v_i, v_j) \in E, \\ 0 & \text{otherwise.} \end{cases}$$

An acyclic digraph H is *Bott equivalent* to an acyclic digraph D if H has an isomorphic digraph that is obtained from D by successively applying *local complementations* and *slides*. In the following, Choi and Oum[10] introduced local complementations and slides.

For a vertex v of a digraph D, let $D * v$ be a digraph obtained by adding an arc (u, w) if $(u, w) \notin E$, or removing the arc (u, w) otherwise, for each pair $(u, w) \in N_D^-(v) \times N_D^+(v)$ with $u \neq w$. This operation to obtain $D * v$ from D is called a *local complementation* at v. A local complementation on digraphs was first introduced by Bouchet[1].

For two sets X and Y, we write $X \triangle Y = (X \setminus Y) \cup (Y \setminus X)$. For two distinct vertices v, w having the same set of in-neighbors in a digraph D, we define $D \triangle vw$ to be a digraph obtained by replacing $N_D^+(w)$ with $N_D^+(w) \triangle N_D^+(v)$. This operation to obtain $D \triangle vw$ from D is called a *slide*.

It is easy to observe that if D is an acyclic digraph, then so are $D * v$ and $D \triangle xy$ (assuming $N_D^-(x) = N_D^-(y)$).

If D is an acyclic digraph, then its adjacency matrix A_D with respect to its acyclic ordering is an upper-triangular square matrix with zero diagonals, i.e, A_D is a Bott matrix. So, there is a bijection from the set of Bott matrices of size n to the set of acyclic digraphs on vertices $\{v_1, \dots, v_n\}$. Therefore we can study real Bott manifolds in terms of acyclic digraphs.

Theorem 2.1 ([10], Theorem 4.4)**.** *Two acyclic digraphs D and H are Bott equivalent if and only if the corresponding real Bott manifolds $M(A_D)$ and $M(A_H)$ are diffeomorphic.*

3 Anosov diffeomorphisms on real Bott manifolds and acyclic digraphs

In this section, we study the relation between an Anosov diffeomorphism of a real Bott manifold $M(A)$ and an acyclic digraph D corresponding to a Bott matrix A as the adjacency matrix of D.

Given an n-RBM $M(A) = T^k \times_{(\mathbb{Z}_2)^s} M(B)$. Let $\{v_1, \dots, v_n\}$ be the set of vertices of an acyclic diagraph D corresponding to the adjacency matrix A.

By definition, if an entry $a_{ij} = 1$ in A then there is an arc $(v_i, v_j) \in E(D)$. This means, $v_i \in N_D^-(v_j)$ and $v_j \in N_D^+(v_i)$.

Since each Bott matrix A can be reduced to (1), $N_D^-(v_i) = \emptyset$ for $i = 1, \ldots, k$ and $N_D^-(v_j) \neq \emptyset$ for $j = k+1, \ldots, n$. Note that $\{v_{k+1}, \ldots, v_n\}$ is the set of vertices of an acyclic diagraph H corresponding to the adjacency matrix B.

Theorem 3.1. *Given a real Bott manifold $M(A)$, and let D be the acyclic digraph corresponding to the adjacency matrix A. $M(A)$ supports an Anosov diffeomorphism if and only if for each $v \in V(D)$ there is a $w \in V(D)$ ($v \neq w$) such that $N_D^-(v) = N_D^-(w)$.*

Proof. Let F be the holonomy group of $M(A)$ and $T\colon F \to GL(n, \mathbb{Z})$ be the associated holonomy representation. It is easy to see that the holonomy representation T of $M(A)$ has the diagonal form:

$$T(x) = \left(\begin{array}{c|ccc} I_k & & 0 & \\ \hline & \hat{1} & & \\ 0 & & \ddots & \\ & & & \hat{1} \end{array}\right), \quad \forall x \in F, \quad \hat{1} \in \{1, -1\}.$$

The diagonal entries of $T(x)$ correspond to the entries in a row of A where the entries 1 and -1 in $T(x)$ correspond to 0 and 1 respectively in a row of A. By Maschke's theorem[11], since F is finite, the holonomy representation T is completely reducible over $\mathbb{Q}$ and so T decomposes as a direct sum of $\mathbb{Q}$-irreducible components:

$$T(x) = T_1(x) \oplus \cdots \oplus T_n(x) \quad \forall x \in F,$$
$$T_i(x) \in \{1, -1\} \quad \forall i = 1, \ldots, n.$$

So each $\mathbb{Q}$-irreducible component T_i of T has dimension one. Obviously, such components are not reducible over $\mathbb{R}$. By definition, T_i is equivalent to T_j ($T_i \sim T_j$, $i \neq j$), if there exists $m \in GL(1, \mathbb{Z})$ such that $mT_i(x) = T_j(x)m$, $\forall x \in F$.

($\Rightarrow$) Assume that $M(A)$ admits an Anosov diffeomorphism. Since each $\mathbb{Q}$-irreducible component T_i is not reducible over $\mathbb{R}$, by Theorem 1.2, there is no $\mathbb{Q}$-irreducible component of T of multiplicity one (i.e., each 1-dimensional $\mathbb{Q}$-irreducible component of T has at least multiplicity two). Hence for each T_i, there is T_j ($i \neq j$) such that $T_i \sim T_j$ (i.e., for each column x of A, there is a column y such that $x = y$). So, in terms of the acyclic digraph D, for each $v \in V(D)$ there is $w \in V(D)$ such that $N_D^-(v) = N_D^-(w)$.

($\Leftarrow$) Assume that $M(A)$ does not support an Anosov diffeomorphism. Again by Theorem 1.2, there is a $\mathbb{Q}$-irreducible component of multiplicity one which is not reducible over $\mathbb{R}$. Since each $\mathbb{Q}$-irreducible component T_i is not reducible over $\mathbb{R}$, there is $v \in V(D)$ such that $N_D^-(v) \neq N_D^-(w)$ for any $w \in V(D)$. So we have a contradiction. □

Ishida [4] proved that an n-RBM $M(A)$ is symplectic if and only if $|\{k|N_D^-(v_k) = N_D^-(v_j)\}|$ is even for every $j \in \{1, \dots, n\}$, with $V(D) = \{v_1, \dots, v_n\}$. This means each symplectic real Bott manifold admits an Anosov diffeomorphism.

Corollary 3.2. *If a real Bott manifold $M(A)$ admits an Anosov diffeomorphism then*

a) The source of D is not unique.

b) For every pair $\{v, w\} \subset V(D)$, $|N_D^+(v) \cap N_D^+(w)| \neq 1$.

c) For every pair $\{v, w\} \subset V(D)$, $|N_D^+(v) \setminus \{N_D^+(v) \cap N_D^+(w)\}| \neq 1$ and $|N_D^+(w) \setminus \{N_D^+(v) \cap N_D^+(w)\}| \neq 1$.

d) For each $v_j \in V(D)$, $|N_D^+(v_j) \setminus \{\bigcup_{\substack{i=1 \\ i\neq j}}^n N_D^+(v_i)\}| \neq 1$.

Proof. Recall that $v \in V(D)$ is a source of D if $deg_D^-(v) = 0$. If the source of D is unique, then the corresponding $M(A)$ admits a maximal S^1-action (i.e., $b_1(M(A)) = 1$). Hence $M(A)$ does not admit an Anosov diffeomorphism ([5]). For b), assume that there exists a pair $\{v, w\} \in V(D)$ such that $|N_D^+(v) \cap N_D^+(w)| = 1$. Let $u \in N_D^+(v) \cap N_D^+(w)$. Then $N_D^-(u) \neq N_D^-(x)$ for any $x \in V(D)$. We have a contradiction. For c) and d), the proofs are similar to b). $\square$

Since each Bott matrix A can be reduced to (1), as a consequence we obtain the following corollary.

Corollary 3.3. *If an n-RBM $M(A) = T^k \times_{(\mathbb{Z}_2)^s} M(B)$ admits an Anosov diffeomorphism then the $(n-k)$-dimensional real Bott manifold $M(B)$ also admits an Anosov diffeomorphism.*

This corollary coinsides with Nazra's result (Theorem 2.1 in [3]) which is proved by using topological method.

In order to determine the number of distinct diffeomorphism classes of $M(A)$ supporting Anosov diffeomorphisms, we can apply Theorem 3.1 together with Theorem 2.1.

Example 3.1. *Here we give an example that if a real Bott manifold $M(B)$ admits an Anosov diffeomorphism then a real Bott manifold $M(A) = T^k \times_{(\mathbb{Z}_2)^s} M(B)$ obtained from $M(B)$ by $(\mathbb{Z}_2)^s$-action does not necessarily admit an Anosov diffeomorphism.*

Consider the 2-dimensional torus T^2. The corresponding Bott matrix is $B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. It is well known that every diffeomorphism on T^n $(n \geq 2)$ admits an Anosov diffeomorphism ([12]).

Now we obtain a 4-dimensional real Bott manifold $M(A)$ with $(\mathbb{Z}_2)^2$-action on $M(B)$ where the corresponding Bott matrix $A = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$. *By Theorem 3.1, it is clear that $M(A)$ does not admit an Anosov diffeomorphism.*

4 Conclusion

Related to the acyclic digraph which correspond to a real Bott manifold, we obtain necessary and sufficient conditions for the real Bott manifold to admit an Anosov diffeomorphism.

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