

DIFFERENTIAL POLYNOMIAL RINGS WHICH ARE GENERALIZED
ASANO PRIME RINGS

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Let $R[x; \delta]$ be a differential polynomial ring over a prime Goldie ring R in an indeterminate x , where δ is a derivation of R . In this paper, we describe explicitly the group of δ -stable v - R -ideals and using this results, we show that $R[x; \delta]$ is a generalized Asano prime ring if and only if R is a δ -generalized Asano prime ring.

Key words : Differential polynomial ring; generalized Asano prime ring; prime Goldie ring; δ -stable v -ideal.

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1. INTRODUCTION

Throughout this paper, R denotes a prime Goldie ring with simple Artinian quotient ring Q (in other word, R is an order in a simple Artinian ring Q) and $R[x; \delta]$ is a differential polynomial ring over R in an indeterminate x with multiplication $xa = ax + \delta(a)$, where δ is a derivation of R .

We define the concept of δ -Krull prime rings and prove that $R[x; \delta]$ is a Krull prime ring if and only if R is a δ -Krull prime ring. We also determine the set of all maximal v -ideals of $R[x; \delta]$ when it is a Krull prime ring.

In [8], we defined a notion of a σ -generalized Asano prime ring motivated by [1] and [2] and proved that a skew polynomial ring $R[x; \sigma]$ over R in an indeterminate x is a generalized Asano prime ring if and only if R is a σ -generalized Asano prime ring, where σ is an automorphism of R .

In this paper, we define a notion of δ -generalized Asano prime rings that are δ -Krull prime rings whose δ - v -ideals are invertible. We obtain that $R[x; \delta]$ is a generalized Asano prime ring if and only if R is a δ -generalized Asano prime ring, which is proved by using the complete description of maximal v -ideals of $R[x; \delta]$.

We refer readers to [9] or [10] for details of maximal orders and R -ideals.

2. δ -GENERALIZED ASANO PRIME RINGS

First we introduce some notation. For any right (left) R -ideals I (J), let

$$(R : I)_l = \{q \in Q \mid qI \subseteq R\} \quad ((R : J)_r = \{q \in Q \mid Jq \subseteq R\})$$

which is a left (right) R -ideal and

$$I_v = (R : (R : I)_l)_r \quad ({}_vJ = (R : (R : J)_r)_l)$$

which is a right (left) R -ideal containing I (J). I (J) is called a right (left) v -ideal if $I_v = I$ (${}_vJ = J$). In case I is a two-sided R -ideal, it is said to be a v -ideal if $I_v = I = {}_vI$.

Let $E(Q/R)$ be the injective hull of a right R -module Q/R and let $F(\tau)$ be a right Gabriel topology corresponding to the torsion theory cogenerated by $E(Q/R)$. Then

$$F(\tau) = \{H : \text{right ideal of } R \mid (R : r^{-1}H)_l = R \text{ for any } r \in R\}$$

where $r^{-1}H = \{a \in R \mid ra \in H\}$ by [11, Proposition 5.5, p.147]. For a right ideal I of R , let

$$\text{cl}(I) = \{r \in R \mid rH \subseteq I \text{ for some } H \in F(\tau)\}.$$

If $\text{cl}(I) = I$, then I is said to be τ -closed. We note that $I \subseteq \text{cl}(I) \subseteq I_v$ for any right R -ideal of R and so right v -ideals are τ -closed. R is called a right τ -Noetherian ring if R satisfies the a.c.c. on τ -closed right ideals. Similarly we define τ -closed left ideals and left τ -Noetherian rings and R is said to be τ -Noetherian if it is right and left τ -Noetherian. We note that if R is right or left τ -Noetherian, then R satisfies the a.c.c. on right or left v -ideals of R .

Let δ be a derivation of R , that is, $\delta(ab) = \delta(a)b + a\delta(b)$ for $a, b \in R$. Then δ is extended to a derivation of Q by $\delta(ac^{-1}) = \delta(a)c^{-1} - ac^{-1}\delta(c)c^{-1}$, where $a \in R$ and c is a regular element of R .

An R -ideal I is called a δ -stable ideal (or a δ -ideal for short) if $\delta(I) \subseteq I$. R is called a δ -Krull prime ring if it is a τ -Noetherian and a δ -maximal order in Q , that is, $O_l(A) = R = O_r(A)$ for any δ -ideal A of R , where

$$O_l(A) = \{q \in Q \mid qA \subseteq A\} \text{ and } O_r(A) = \{q \in Q \mid Aq \subseteq A\}.$$

In case $\delta = 0$, R is said to be a Krull prime ring. Then we have the following.

Proposition 2.1 — R is a δ -Krull prime ring if and only if $S = R[x; \delta]$ is a Krull prime ring.

PROOF : By [9, Proposition 2.3.15], R is τ -Noetherian if and only if $R[x; \delta]$ is τ -Noetherian. Suppose R is a δ -Krull prime ring and let A be a non-zero ideal of

S . Then $A' = AT$, where $T = Q[x; \delta]$, is an ideal of T by [6, Theorem 9.20 (a)]. Let $q \in O_l(A)$, that is $qA \subseteq A$. Then $qA' \subseteq A'$ and $q \in O_l(A') = T$ since T is a principal ideal ring. Write $q = q_n x^n + \cdots + q_1 x + q_0$, where $q_i \in Q$ and let

$$L(A) = \{a_l \in R \mid \exists f(x) = a_l x^l + \cdots + a_1 x + a_0 \in A\},$$

which is a δ -ideal of R . For any $a \in L(A)$, there is an $a(x) = ax^l + \cdots + a_1 x + a_0 \in A$ and $qa(x) \in A$ implies $q_n a \in L(A)$, that is $q_n L(A) \subseteq L(A)$. Hence $q_n \in R$. Continuing this method, we have $q \in S$ and similarly $O_r(A) = S$. Hence S is a maximal order and so it is a Krull prime ring.

Conversely suppose that S is a Krull prime ring and let \mathfrak{a} be a δ -ideal of R . Then $A = \mathfrak{a}[x; \delta]$ is an ideal of S . Let $q \in O_l(\mathfrak{a})$. Then $qA \subseteq A$ and so $q \in S \cap Q = R$. Similarly $O_r(\mathfrak{a}) = R$ since $S = R[x; -\delta] = \{x^n a_n + \cdots + a_1 x + a_0 \mid a_i \in R\}$. Hence R is a δ -maximal order and so it is a δ -Krull prime ring. \square

If R is a δ -Krull prime ring, then for any δ -ideal A ,

$$(R : A)_l = A^{-1} = \{q \in Q \mid AqA \subseteq A\} = (R : A)_r$$

and so $A_v = A^{-1-1} = {}_v A$ follows.

Let $D_\delta(R)$ be the set of all δ -v-ideals. For $A, B \in D_\delta(R)$, we define $A \circ B = (AB)_v$. The following lemma is proved in the standard way ([9, Theorem 2.1.2]). However we give a complete proof for reader's convenience.

Lemma 2.2 — If R is a δ -Krull prime ring, then $D_\delta(R)$ is a free abelian group generated by maximal δ -v-ideals of R with multiplication \circ .

PROOF : Let \mathfrak{m} be any maximal δ -v-ideal of R . Then \mathfrak{m} is v-invertible, that is $\mathfrak{m}^{-1} \circ \mathfrak{m} = R = \mathfrak{m} \circ \mathfrak{m}^{-1}$ because $R \supseteq \mathfrak{m}^{-1} \mathfrak{m} \supseteq \mathfrak{m}$ and $R \supseteq \mathfrak{m} \mathfrak{m}^{-1} \supseteq \mathfrak{m}$.

For any different maximal δ -v-ideals \mathfrak{m} and \mathfrak{n} of R , we claim that $\mathfrak{m} \circ \mathfrak{n} = \mathfrak{m} \cap \mathfrak{n} = \mathfrak{n} \circ \mathfrak{m}$. From $\mathfrak{m} \supseteq \mathfrak{a} = \mathfrak{m} \cap \mathfrak{n} \supseteq \mathfrak{m} \mathfrak{n}$, we have $R \supseteq \mathfrak{m}^{-1} \circ \mathfrak{a} \supseteq {}_v(\mathfrak{m}^{-1} \mathfrak{m} \mathfrak{n}) = \mathfrak{n}$ and so $\mathfrak{m}^{-1} \circ \mathfrak{a} = \mathfrak{n}$. Thus $\mathfrak{a} = \mathfrak{m} \circ \mathfrak{n}$ and similarly $\mathfrak{a} = \mathfrak{n} \circ \mathfrak{m}$.

Next we will show that any δ -v-ideal of R is a finite product of maximal δ -v-ideals of R . Assume, on the contrary, that there is a δ -v-ideal \mathfrak{a} of R which is not

a finite product of maximal δ - v -ideals of R . Choose \mathfrak{a} maximal with this property. Then \mathfrak{a} is not a maximal δ - v -ideal of R . Let \mathfrak{m} be a maximal δ - v -ideal of R with $\mathfrak{m} \supset \mathfrak{a}$. Then $R \supseteq \mathfrak{m}^{-1} \circ \mathfrak{a} \supset \mathfrak{a}$ and so $\mathfrak{m}^{-1} \circ \mathfrak{a}$ is a finite product of maximal δ - v -ideals of R . Thus so is \mathfrak{a} , which is a contradiction.

Finally let \mathfrak{a} be any δ - v -ideal and $\mathfrak{b} = \{r \in R \mid r\mathfrak{a} \subseteq R\}$. Then \mathfrak{b} is a δ - v -ideal of R and so \mathfrak{b} and $\mathfrak{b} \circ \mathfrak{a}$ are both finite product of maximal δ - v -ideals of R . Hence $\mathfrak{a} = \mathfrak{b}^{-1} \circ (\mathfrak{b} \circ \mathfrak{a})$ is a finite product of maximal δ - v -ideals \mathfrak{m} of R or its inverse \mathfrak{m}^{-1} . Thus $D_\delta(R)$ is a free abelian group generated by maximal δ - v -ideals.

A δ -ideal P of R is called a δ -prime ideal if $IJ \subseteq P$ for δ -ideals I and J of R , then $I \subseteq P$ or $J \subseteq P$.

Proposition 2.3 — Suppose $S = R[x; \delta]$ is a Krull prime ring. Then $\{\mathfrak{m}[x; \delta], M \mid \mathfrak{m} \text{ is a maximal } \delta - v - \text{ideal of } R \text{ and } M = M' \cap S, \text{ where } M' \text{ is a maximal ideal of } T = Q[x; \delta]\}$ is the set of all maximal v -ideals of S .

PROOF : Let M be any maximal v -ideal of S and $\mathfrak{m} = M \cap R$. It is easy to see that M is a prime ideal and \mathfrak{m} is a δ -prime v -ideal.

(i) In case $\mathfrak{m} \neq (0)$. By [7, Lemma 1.3], $\mathfrak{m}[x; \delta]$ is a prime ideal and it is a v -ideal (see the proof of [12, Lemma 3]). Hence $M = \mathfrak{m}[x; \delta]$ and \mathfrak{m} is a maximal δ - v -ideal since M is a minimal prime ideal of S by [10, Proposition 5.1.9]. Conversely suppose \mathfrak{m} is a maximal δ - v -ideal. Then it is a δ -prime ideal and so $\mathfrak{m}[x; \delta]$ is a prime v -ideal. Hence it is a maximal v -ideal.

(ii) In case $\mathfrak{m} = (0)$. Put $M' = MT$, a proper ideal of T . To prove that M' is a maximal ideal of T , let N' be a maximal ideal of T with $N' \supseteq M'$. Then $N' = NT$, where $N = N' \cap S$. Since S is τ -Noetherian and T is a principal ideal ring, it follows that

$$\begin{aligned} N' &= N'_v = (T : (T : N')_l)_r = (T : T(S : N)_l)_r \\ &= (S : (S : N)_l)_r T = N_v T. \end{aligned}$$

Thus N is a v -ideal and so $N = M$. Hence M' is a maximal ideal of T with $M = M' \cap S$.

Conversely if M' is a maximal ideal of T , then $M = M' \cap S$ is a maximal v -ideal which is clear from the proof above. \square

Lemma 2.4 — Suppose $S = R[x; \delta]$ is a Krull prime ring. If A is a v - S -ideal contained in $T = Q[x; \delta]$ with $\mathfrak{a} = A \cap Q \neq (0)$, then $A = \mathfrak{a}[x; \delta]$.

PROOF : By Lemma 2.2 and Proposition 2.3, $A = \mathfrak{b}[x; \delta] \circ B$ where \mathfrak{b} is a δ - v -ideal and $B = (M_1^{e_1} \cdots M_k^{e_k})_v$ such that $M'_i = M_i T$ is a maximal ideal of T , $1 \leq i \leq k$. Thus $T = AT = (\mathfrak{b}[x; \delta]B)_v T = BT$ and so $e_1 = \cdots = e_k = 0$. Hence $A = \mathfrak{b}[x; \delta]$ with $\mathfrak{b} = A \cap Q = \mathfrak{a}$. \square

An R -ideal A is said to be invertible if $A^{-1}A = R = AA^{-1}$. Then we call R a δ -generalized Asano prime ring if it is a δ -Krull prime ring whose δ - v -ideals are invertible. In case $\delta = 0$, R is said to be a generalized Asano prime ring. Let R be a δ -Krull prime ring. Then, by Lemma 2.2, R is a δ -generalized Asano prime ring if and only if any δ - v -ideal of R is invertible, which is equivalent to that any maximal δ - v -ideal of R is invertible.

Lemma 2.5 — Suppose R is a δ -generalized Asano prime ring. If $B = (M_1^{e_1} \cdots M_k^{e_k})_v$ is a v -ideal of $S = R[x; \delta]$, where $M'_i = M_i T$ is a maximal ideal of T , then B is invertible.

PROOF : Since BT is an ideal, $BT = wT$ for some central element w in T by [3, Remark 1, p.95] and so $w^{-1}B$ is a v - S -ideal such that $w^{-1}B \subseteq T$ with $\mathfrak{a} = w^{-1}B \cap Q \neq (0)$. It follows that from Lemma 2.4 that $w^{-1}B = \mathfrak{a}[x; \delta]$. Hence $B = w\mathfrak{a}[x; \delta]$ is invertible. \square

Now we obtain the main result of this paper.

Theorem 2.6 — *Let R be a prime Goldie ring in a simple Artinian ring Q . Then R is a δ -generalized Asano prime ring if and only if $S = R[x; \delta]$ is a generalized Asano prime ring.*

PROOF : Suppose R is a δ -generalized Asano prime ring. Let A be a v -ideal of S . Then $A = \mathfrak{a}[x; \delta] \circ B = \mathfrak{a}[x; \delta]B$, where \mathfrak{a} is a δ - v -ideal and B is as in Lemma 2.5. Hence A is invertible and so S is a generalized Asano prime ring.

Conversely suppose S is a generalized Asano prime ring. Let \mathfrak{a} be a δ - v -ideal of R . Then $A = \mathfrak{a}[x; \delta]$ is invertible with $A^{-1} = \mathfrak{a}^{-1}[x; \delta]$ and so \mathfrak{a} is invertible. Hence R is a δ -generalized Asano prime ring. \square

In [1] and [2], she studied Noetherian generalized Dedekind prime rings, which are generalized Asano prime rings in our sense. We give an example of non-Noetherian generalized Asano prime rings.

Lemma 2.7 — Let D be a commutative domain with quotient field K and $R = D[\mathbf{x}]$ be a polynomial ring over D in indeterminates $\mathbf{x} = \{x_i\}$ (we do not assume that $\{x_i\}$ is finite). Further let \mathbf{x}' be a finite subset of \mathbf{x} and I be a $D[\mathbf{x}']$ -ideal. Then $(R : I[\mathbf{x} - \mathbf{x}'])_l = (D[\mathbf{x}'] : I)_l[\mathbf{x} - \mathbf{x}']$ and in particular $(I[\mathbf{x} - \mathbf{x}'])_v = I_v[\mathbf{x} - \mathbf{x}']$.

PROOF: It is clear $(D[\mathbf{x}'] : I)_l[\mathbf{x} - \mathbf{x}'] \subseteq (R : I[\mathbf{x} - \mathbf{x}'])_l$. Let α be an element in $(R : I[\mathbf{x} - \mathbf{x}'])_l$ and $K(\mathbf{x}')$ be the quotient field of $K[\mathbf{x}']$. Then $IK(\mathbf{x}') = K(\mathbf{x}')$ and $\alpha I \subseteq R$ imply $\alpha \in \alpha K(\mathbf{x}') = \alpha IK(\mathbf{x}') \subseteq RK(\mathbf{x}') \subseteq K(\mathbf{x}')[\mathbf{x} - \mathbf{x}']$. So we can write $\alpha = q_1\mathbf{x}_1 + \cdots + q_i\mathbf{x}_i + \cdots + q_0$, where $q_i \in K(\mathbf{x}')$ and \mathbf{x}_i 's are subsets of $\mathbf{x} - \mathbf{x}'$. Hence $q_i I \subseteq D[\mathbf{x}']$ for each i and $q_i \in (D[\mathbf{x}'] : I)_l$. Therefore $\alpha \in (D[\mathbf{x}'] : I)_l[\mathbf{x} - \mathbf{x}']$ follows. \square

Lemma 2.8 — Let D be a commutative Krull domain and $R = D[\mathbf{x}]$ be a polynomial ring over D in indeterminates \mathbf{x} , where \mathbf{x} is infinite or finite. Then

(1) R is a Krull domain and it is not Noetherian if \mathbf{x} is an infinite set.

(2) For any maximal v -ideal P of R , there exists a finite subset \mathbf{x}' of \mathbf{x} such that $P = P_0[\mathbf{x} - \mathbf{x}']$, where $P_0 = P \cap D[\mathbf{x}']$ and P_0 is a maximal v -ideal of $D[\mathbf{x}']$.

PROOF: (1) See for example [4, Theorem 4.2.9].

(2) Let P be a maximal v -ideal of $R = D[\mathbf{x}]$. Then there exists a finite subset of \mathbf{x}' of \mathbf{x} with $P_0 = D[\mathbf{x}'] \cap P \neq (0)$. If $AB \subseteq P_0$ for ideals A and B of $D[\mathbf{x}']$, then $A[\mathbf{x} - \mathbf{x}']B[\mathbf{x} - \mathbf{x}'] = AB[\mathbf{x} - \mathbf{x}'] \subseteq P_0[\mathbf{x} - \mathbf{x}'] \subseteq P$ and so we have $A[\mathbf{x} - \mathbf{x}'] \subseteq P$ or $B[\mathbf{x} - \mathbf{x}'] \subseteq P$. Hence $A \subseteq P_0$ or $B \subseteq P_0$ holds and P_0 is a prime ideal of $D[\mathbf{x}']$. Now $P_0[\mathbf{x} - \mathbf{x}'] \subseteq P$ and so $(P_0)_v[\mathbf{x} - \mathbf{x}'] = (P_0[\mathbf{x} - \mathbf{x}'])_v \subseteq P_v = P$ by Lemma 2.7. Hence $(P_0)_v = P_0$ and P_0 is a prime v -ideal. Since P is a minimal

prime ideal, we have $P = P_0[\mathbf{x} - \mathbf{x}']$. □

Example : Let D be a generalized Dedekind domain. Then $R = D[\mathbf{x}]$ is also a generalized Dedekind domain. If \mathbf{x} is infinite, then R is not Noetherian.

PROOF : Since R is a Krull domain, it is sufficient to prove that any maximal v -ideal is invertible. But this follows from the relation $P = P_0[\mathbf{x} - \mathbf{x}']$ because $D[\mathbf{x}']$ is a generalized Dedekind domain. □

A generalized Asano prime ring is a Krull prime ring. But the converse does not necessarily hold (see [5, p.8, Example 1.10]).

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