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On locating-chromatic number of complete n-ary tree

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Abstract

Let c be a vertex k-coloring on a connected graph $G(V,E)$. Let $\Pi = \{C_1, C_2, ..., C_k\}$ be the partition of $V(G)$ induced by the coloring c. *The color code* $c_\Pi(v)$ of a vertex v in G is $(d(v,C_1), d(v,C_2), ..., d(v,C_k))$, where $d(v,C_i) = min\{d(v,x)|x \in C_i\}$ for $1 \leq i \leq k$. If any two distinct vertices u, v in G satisfy that $c_\Pi(u) \neq c_\Pi(v)$, then c is called *a locating k-coloring* of G. *The locating-chromatic number* of G, denoted by $\chi_L(G)$, is the smallest k such that G admits a locating k-coloring. Let $T(n,k)$ be *a complete n-ary tree*, namely a rooted tree with depth k in which each vertex has n children except for its leaves. In this paper, we study the locating-chromatic number of $T(n,k)$.

Keywords: color code, locating-chromatic number, complete n-ary tree.

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1. Introduction

The locating-chromatic number of a graph is a combined concept between the coloring and partition dimension of a graph. The concept of partition dimension of a graph was introduced by Chartrand et al. [8] in 1998, and subsequently the concept of locating-chromatic number of a graph was initiated by Chartrand et al. [9] in 2002.

Let $G = (V,E)$ be a connected graph. Let $\Pi = \{C_1, C_2, ..., C_k\}$ be a k-partition of $V(G)$. The *color code* $c_\Pi(v)$ of vertex v is the ordered k-tuple $(d(v,C_1), d(v,C_2), ..., d(v,C_k))$, where $d(v,C_i) = \min\{d(v,x)|x \in C_i\}$ for $1 \leq i \leq k$. If all vertices of G have distinct color codes, then c is called a *locating k-coloring* of G. The *locating-chromatic number* of G, denoted by $\chi_L(G)$, is the smallest k such that G has a locating k-coloring.

It is clear that the only graph having the locating-chromatic number 1 and 2 is K_1 and K_2, respectively. The only graph of order $n \geq 3$ having the locating-chromatic number n is the complete multipartite graph. Furthermore, Chartrand et al. [10] characterized all graphs of order n with the locating-chromatic number $n-1$. They also gave some conditions of graph G in which $n-2$ is an upper bound of its locating-chromatic number. Recently, Asmiati and Baskoro [1] characterized all graphs containing cycles with locating-chromatic number 3.

Chartrand et al. [9] determined the locating-chromatic numbers of cycles. For graph derived from some graph operations, Baskoro and Purwasih [7] determined the locating-chromatic number for the corona product of two graphs. Behtoei and Ommoomi studied the locating-chromatic number for the Cartesian product of graph [5], the join of graphs [6] and the Kneser graph [4].

For trees, as far as we know, we just have the following results. Chartrand et al. [9] determined the locating-chromatic numbers of paths and double stars. Furthermore, Chartrand et al. [10] showed that for any integer $k \in [3, n]$, and $k \neq n-1$, there exists a tree on n vertices with the locating-chromatic number k. Recently, Asmiati et al. determined the locating-chromatic number of firecrackers [2] and an amalgamation of stars [3]. However, there are many classes of trees whose the locating-chromatic number are still not known. Thus, in this paper, we determine the locating-chromatic number of some particular class of trees, namely a complete n-ary tree.

Let us begin to state the following lemma and corollary which are useful to obtain our main results.

Lemma 1.1. [9] *Let c be a locating coloring in a connected graph G. If u and v are distinct vertices of G such that $d(u,w) = d(v,w)$ for all $w \in V(G) - \{u, v\}$, then $c(u) \neq c(v)$. In particular, if u and v are adjacent to the same vertices, then $c(u) \neq c(v)$.*

Corollary 1.2. [9] *If G is a connected graph containing a vertex adjacent to k leaves, then $\chi_L(G) \geq k+1$.*

2. Main Results

For $n, k \geq 3$, let us denote by $T(n,k)$ a complete n-ary of depth k and each vertex has n children except for its leaves. The *depth* of $T(n,k)$ is the length of a path from its root vertex to its leaves. Therefore, $T(n,1)$ is a star and $T(n,2)$ is a lobster.

A graph $T(n,k)$ can be constructed recursively, namely by connecting the root vertices of the n copies of $T(n,k-1)$ to a new vertex x_0. In this view, The ith copy of $T(n,k-1)$ in $T(n,k)$ is denoted by $T^i(n,k-1)$ and the ith copy of vertex x of $T(n,k-1)$ in $T(n,k)$ is denoted by x^i, for $i = 1, 2, \cdots, n$, see Figure 1.

Now, we will show that $\chi_L(T(n,1)) = \chi_L(T(n,2)) = n+1$, for any $n \geq 2$, as in the following theorem.

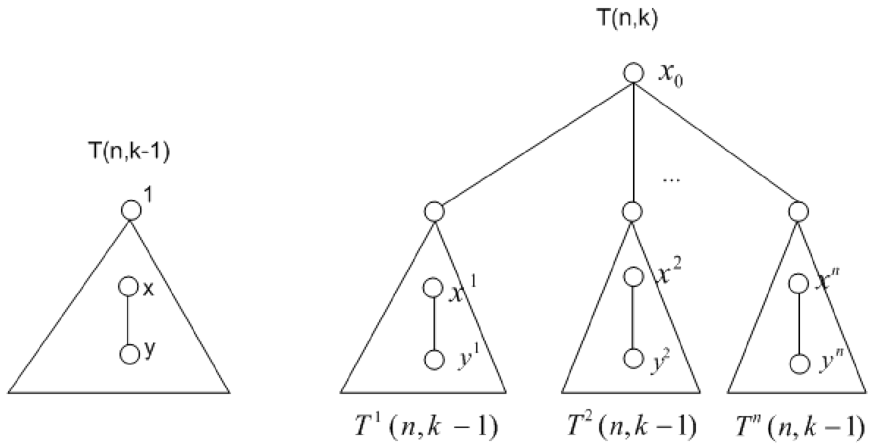

Figure 1: Graph $T(n,k)$.

Theorem 2.1. *If* $n \geq 2$ *then* $\chi_L(T(n,1)) = \chi_L(T(n,2)) = n+1$.

Proof. It is clear that $\chi_L(T(n,1)) = n+1$. Now, let us show that $\chi_L(T(n,2)) = n+1$. Let x_0 be the root vertex of $T(n,2)$. Let $L_1 = \{x_1, x_2, ..., x_n\}$, $L_2 = \{y_{ij} | i, j \in [1,n]\}$ and $V(T(n,2)) = \{x_0\} \bigcup L_1 \bigcup L_2$. To show that $\chi(T(n,2)) \leq n+1$, define a coloring $c : V(T(n,2)) \rightarrow \{1,2,...,n+1\}$ such that

$$\begin{aligned} c(x_0) &= 1, \\ c(x_i) &= i+1, \\ c(D_i) &= [1,n+1] \backslash \{i+1\}, \text{ where } D_i = \{y_{ij} | j \in [1,n]\}. \end{aligned}$$

Let $\Pi = \{C_1, C_2, \cdots, C_{n+1}\}$ be the partition of $V(T(n,2))$ induced by c, where C_i is the set of all vertices receiving color i. Next, we will show that the color codes of all vertices are distinct. Let u and v be two distinct vertices with $c(u) = c(v)$. Now, we consider the following cases:

Case 1. $u = x_0$, $v \in L_2$.

If $v = y_{ir}$ for some i and r then $d(u,C) = 1$ and $d(v,C) = 2$ for either $C = C_{i-1}$ or $C = C_{i+1}$. Therefore, $c_\Pi(u) \neq c_\Pi(v)$.

Case 2. $u \in L_1$, $v \in L_2$.

If $u = x_i$ and $v = y_{jr}$, for some i , j , r and $i \neq j$ then $d(u,C) = 1$ and $d(v,C) = 2$ for either $C = C_{i-1}$ or $C = C_{i+1}$. Therefore, $c_\Pi(u) \neq c_\Pi(v)$.

Case 3. $u, v \in L_2$.

If $u = x_{ir}$ and $v = x_{js}$, for some i, j, r, s and $i \neq j$ then $d(u, C_{i+1}) = 1$ and $d(v, C_{i+1}) = 2$. Therefore, $c_\Pi(u) \neq c_\Pi(v)$.

Therefore, all vertices have distinct color codes, and so $\chi_L(T(n,2)) \leq n+1$. By Lemma 1.1, we obtain that $\chi_L(T(n,2)) = n+1$. □

Theorem 2.2. *If* $n \geq 3$ *then* $\chi_L(T(n,3)) = n+2$.

Proof. Let x_0 be the root vertex of $T(n,3)$. For $i = 1,2,3$ define $L_i = \{v \in T(n,3) | d(v, x_0) = i\}$. Let $V(T(n,3)) = \{x_0\} \cup L_1 \cup L_2 \cup L_3$. Let c be a locating $(n+1)$-coloring of $T(n,2)$, as defined in Theorem 2.1. For $j = 1, 2, \cdots, n$, let $T^j(n,2)$ be the j th copy of $T(n,2)$ in $T(n,3)$ is denoted by $T^j(n,k-1)$ and x_j be the j th copy of vertex x of $T(n,k-1)$ in $T(n,k)$. To show that $\chi(T(n,3)) \leq n+2$, define a new coloring $c^* : V(T(n,3)) \rightarrow \{1, 2, ..., n+2\}$ such that:

$$\begin{aligned} c^*(x^i) &= (c(x) + (i-1)) \bmod n+2, \text{ for } 1 \leq i \leq n, \\ c^*(x_0) &= n+2. \end{aligned}$$

Let $\Pi^* = \{C_1, C_2, ..., C_{n+2}\}$ be the $n+2$-partition of $V(T(n,3))$ induced by c^*, where C_i is the set of all vertices of color i. By the definition of coloring c^* of $T(n,3)$, we obtain that $c^*(x_0) = n+2$, the set of the colors of all vertices in $T^1(n,2)$ is $c^*(T^1(n,2)) = [1, n+1]$ and the set of the colors of all vertices in $T^i(n,2)$ is $c^*(T^i(n,2)) = [1, n+2] \setminus \{i-1\}$, for any $i \in [2, n]$. Furthermore, $c^*(L_1) = [1, n]$, see Figure 2. Next, we will show that the color codes of all vertices in $T(n,3)$ are distinct. Let u and v be two distinct vertices with $c^*(u) = c^*(v)$. If one of $\{u, v\}$ is x_0 then it is clear that $c_{\Pi^*}(u) \neq c_{\Pi^*}(v)$. Now, if none of them is the root vertex then consider the following cases:

Case 1. $u \in L_a$, $v \in L_b$, and $a \neq b$.

Let $a < b$. If $u, v \in V(T^i(n,2))$ for some $i \in [1, n]$ then $d(u, C_{i-1 \bmod n+2}) < d(v, C_{i-1 \bmod n+2})$. Thus, $c_{\Pi^*}(u) \neq c_{\Pi^*}(v)$. If $u \in V(T^i(n,2))$, $v \in V(T^j(n,2))$ for some $i < j$ then $d(u, C_{j-1 \bmod n+2}) < d(v, C_{j-1 \bmod n+2})$. Thus, $c_{\Pi^*}(u) \neq c_{\Pi^*}(v)$.

Case 2. $u, v \in L_a$.

Since all colors of the vertices in L_1 are different then $a = 2$ or 3. If $u, v \in T^i(n,2)$ for some i then $a = 3$ and $c_{\Pi^*}(u) \neq c_{\Pi^*}(v)$, since c is a locating coloring in $T(n,2)$. Let $u \in T^i(n,2)$, $v \in T^j(n,2)$, and $i \neq j$. Then, one of $\{i, j\}$ is not equal to 1. So, we can assume that $j \neq 1$. Thus, $d(v, C_{j-1}) > d(u, C_{j-1})$. This implies that $c_{\Pi^*}(u) \neq c_{\Pi^*}(v)$.

Therefore, in any case, all color codes of the vertices are different, and so $\chi_L(T(n,3)) \leq n+2$. Since, there are more than $n+1$ vertices having n pendants then $\chi_L(T(n,3)) \geq n+2$. This concludes that $\chi_L(T(n,3)) = n+2$. □

Theorem 2.3. *If* $n, k \geq 3$ *then* $\chi_L(T(n,k)) \leq n+k-1$.

Proof. Let x_0 be the root vertex of $T(n,k)$. For $i = 1, 2, \cdots, k$, define $L_i = \{v \in T(n,k) | d(v, x_0) = i\}$. Let $V(T(n,k)) = \{x_0\} \bigcup_{i=1}^{k} L_i$. We are going to prove this theorem by induction on k. For $k = 3$, the theorem holds by Theorem 2.2. Now, assume that the theorem holds for all $l < k$. This means that there is a locating $(n+k-2)$-coloring of $T(n, k-1)$. Now, we are going to show that there exists a locating $(n+k-1)$-coloring of $T(n,k)$.

Let c be a locating $(n+k-2)$-coloring of $T(n, k-1)$ with the color of the root is 1. This coloring is always available by the recursive definition of the coloring as in the proof of Theorem 2.2. By Theorem 2.2, we have a locating $(n+2)$-coloring of $T(n,3)$ with $c(x_0) = n+2$. Then, add all the colors by 1 (in modulo $n+2$) to have a desired locating coloring of $T(n,3)$ with $c(x_0) = 1$, and $c(L_1) = [2, n+1]$. Next, use this coloring and the definition of c^* as in the proof of Theorem 2.2 to construct a $(n+3)$-coloring of $T(n,4)$ with $c(x_0) = n+3$. Then, again add all the colors by 1 (in modulo $n+2$) to have a desired coloring of $T(n,4)$ with $c(x_0) = 1$, and $c(L_1) = [2, n+1]$, and so forth. Assume such $(n+k-2)$-coloring of $T(n, k-1)$ is locating. We shall prove that a coloring $c^* : V(T(n,k)) \to \{1, 2, ..., n+k-1\}$ such that:

$$\begin{aligned} c^*(x^i) &= (c(x) + (i-1)) \bmod n+k-1, \text{ for } 1 \le i \le n, \\ c^*(x_0) &= n+k-1, \end{aligned}$$

is also a locating coloring of $T(n,k)$.

Let $\Pi^* = \{C_1, C_2, ..., C_{n+k-1}\}$ be the $(n+k-1)$-partition of $V(T(n,k))$ induced by c^*, where C_i is the set of all vertices of color i. It is clear that $c^*(x_0) = n+k-1$, the set of the colors of all vertices in $T^1(n, k-1)$ is $[1, n+k-2]$ and the set of the colors of all vertices in $T^i(n, k-1)$ is $[1, n+k-1] \setminus \{i-1\}$ for $2 \le i \le n$. Furthermore, $c^*(L_1) = [1, n]$, see Figure 2. Next, we will show that the color codes of all vertices in $T(n,k)$ are distinct. Let u and v be two distinct vertices with $c^*(u) = c^*(v)$. If one of $\{u, v\}$ is x_0 then it is clear that $c_{\Pi^*}(u) \neq c_{\Pi^*}(v)$. Now, if none of them is the root vertex then consider the following cases:

Case 1. $u \in L_a$, $v \in L_b$, and $a \neq b$.

Let $a < b$. If $u, v \in V(T^i(n, k-1))$ for some $i \in [1, n]$ then let u, v be the ith copies of two vertices $x, y \in V(T(n, k-1))$, respectively. Since $c^*(u) = c(x) + (i-1) \bmod n+k-1$, $c^*(v) = c(y) + (i-1) \bmod n+k-1$, and $c^*(u) = c^*(v)$, then $c(x) = c(y)$. Since c is a locating-coloring in $T(n, k-1)$ then $c_{\Pi}(u) \neq c_{\Pi}(v)$. This implies that $c_{\Pi^*}(u) \neq c_{\Pi^*}(v)$. If $u \in V(T^i(n, k-1))$, $v \in V(T^j(n, k-1))$ for some $i < j$ then $d(u, C_{j-1 \bmod n+k-1}) < d(v, C_{j-1 \bmod n+k-1})$. Thus, $c_{\Pi^*}(u) \neq c_{\Pi^*}(v)$.

Case 2. $u, v \in L_a$.

Since all colors of the vertices in L_1 are different then $a = 2, 3, \cdots, k$. If $u, v \in T^i(n, k-1)$ for some i then let u, v be the ith copies of two vertices $x, y \in V(T(n, k-1))$, respectively. Since $c^*(u) = c(x) + (i-1) \bmod n+k-1$, $c^*(v) = c(y) + (i-1) \bmod n+k-1$, and $c^*(u) = c^*(v)$, then $c(x) = c(y)$. Since c is a locating-coloring in $T(n, k-1)$ then

$c_\Pi(u) \neq c_\Pi(v)$. This implies that $c_{\Pi^*}(u) \neq c_{\Pi^*}(v)$. Now, let $u \in T^i(n,2)$, $v \in T^j(n,2)$, and $i \neq j$. Then, one of $\{i,j\}$ is not equal to 1. So, we can assume that $j \neq 1$. Thus, $d(v, C_{j-1}) > d(u, C_{j-1})$. This implies that $c_{\Pi^*}(u) \neq c_{\Pi^*}(v)$.

Therefore, in any case, all color codes of the vertices are different, and so $\chi_L(T(n,k)) \leq n+k-1$. $\square$

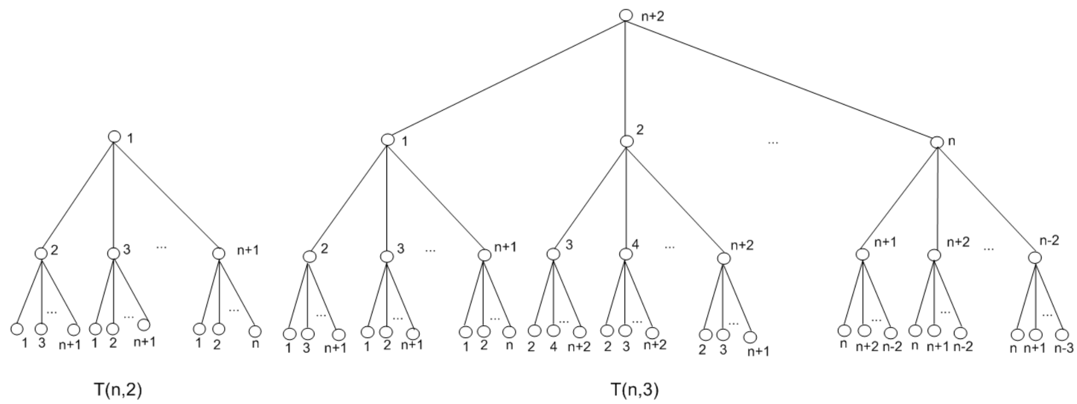

Figure 2: The locating coloring of $T(n,2)$ and $T(n,3)$

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