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Variational approximations for twisted solitons in a parametrically driven discrete nonlinear Schrödinger equation

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Abstract. Intersite lattice solitons with out-of-phase oscillation,or the so-called twisted localized modes, in a parametrically driven discrete nonlinear Schrödinger equation are studied by means of a variational approximation. The proposed ansatz is of the form an exponential function which can approximate solitons in the vicinity of the anticontinuum limit. The approximated solitons resulted from the variational approach are shown in good agreement with the corresponding numerical results.

1. Introduction

The study of solitons in discrete nonlinear Schrödinger (DNLS) equations has attracted a great deal of attention for many researchers. This interest came along with the fact that the DNLS equations arise in a huge number of useful and promising applications. These include optical beams in waveguide arrays [1], Bose-Einstein condensates trapped in a periodic potential [2], and nonlinear resonator arrays in micro- and nano-electromechanical systems (MEMS and NEMS) [3], just to mention a few examples. A comprehensive review of theoretical and experimental researches in the DNLS equations has been well explained in [4].

Due to its nonintegrability, one may expect to develop reasonable methods to approximate discrete soliton solutions in the DNLS model. One of the methods which is well known and has been long used to approximate solutions (including the localized states) of a nonlinear evolution equation is the so-called *variational approximation* (VA). Formulation of this method is based on theory of Lagrangian and Hamiltonian mechanics (see, e.g., [5]) and systematically described in the following steps [6]:

1. Formulate the Lagrangian of the governing equation.
2. Propose a reasonable trial function (ansatz) which contains a finite number of parameters (called variational parameters).
3. Substitute the proposed ansatz into the Lagrangian and evaluate the resulting sums (for discrete systems) or integrations (for continuous system).
4. Find the critical points of the variational parameters by solving the corresponding Euler-Lagrange equations.

   In the context of DNLS model with cubic nonlinearity, the method of VA has been employed in a number of papers and its agreement has been confirmed with the corresponding numerical results; see, e.g., [7-9]. In those references, various trial functions have been exploited where some better than others. For example, the ansatz used in [7] can only construct theone-exited site(onsite) solutions, while the ansatz in [8] is also applicable for the symmetrictwo-exited site(intersite) configurations. Moreover, the proposed ansatz in [9] can predict the asymmetric intersite solutions, thus the VA can also be used to explain bifurcations linking solutions. The validity of the VA for the cubic DNLS equation in the limit of small coupling has been justified rigorously in [10]. In the latter reference, it was shown that the trial function for stationary discrete solitons with more parameters provides more accurate approximations. To explore the study of VA in various applications, one can refer to the review given by Malomed [11].

One of variant of cubic DNLS equation is the one with the inclusion of parametric driving. The existence and stability of such a system have been previously considered in [12,13] using a perturbation expansion, i.e., followed the idea of the anti-continuum (AC) limit approach introduced initially by MacKay and Aubry [14]. In this paper, we study the application of VA for another intersite configuration with out-of-phase oscillation, or the so-called *twisted solitons*, in the cubic DNLS equation under the effect of parametric driving. The variational-based analysis will be particularly performed for the case of the AC limit. Thus we need to choose a reasonable ansatz for such a purpose. In the absence of parametric driving, twisted solitons in the cubic DNLS equation have been considered, e.g.,[15].

Our presentation in the present work is organized as follows. In Section 2, we present the considered model equation and discuss the preliminary analyses of the model. Next, in Section 3 we derive the VA for the steady-state localized solutions of the driven system. The analytical approach is developed such that the approximated solutions can deal with the AC limit case. To check the validity of the VA, in Section 4 we give comparisons between the analytical findings and the corresponding numerical results. Finally, in Section 5 we summarize our results and address some interesting problems for future study.

2. The Model Equation

We consider a parametrically driven discrete nonlinear Schrödinger (PDNLS) equation in the focusing nonlinearity, given by

$$\mathrm{i}\dot{\phi}_n = -\varepsilon\Delta_2\phi_n + \Lambda\phi_n + \gamma\bar{\phi}_n - |\phi_n|^2\phi_n. \quad (1)$$

In the above equation, $\phi_n \equiv \phi_n(t)$ is a complex-valued wave function of time $t$at site $n \in \mathbb{Z}$, the overdot and the overline denote, respectively, the time derivative and the complex conjugation, $\varepsilon > 0$ represents the coupling constant between two adjacent sites, $\Delta_2\phi_n = \phi_{n+1} - 2\phi_n + \phi_{n-1}$ is the discrete Laplacian in one spatial dimension, and γ is the parametric driving coefficient with frequency Λ. To the best of our knowledge, Eq. (1) was studied initially by Hennig [16] with the inclusion of a damping term.

In the undriven case ($\gamma = 0$), Eq. (1) admits the existence of the so-called *phase* or *gauge* invariance [16,17], i.e., if we use a transformation $\phi_n \to \phi_n \mathrm{e}^{\mathrm{i}\theta}$,for an arbitrary $\theta \in \mathbb{R}$, then Eq. (1) is left unchanged. In the driven case ($\gamma \neq 0$), this invariance no longer exists due to the presence of the complex conjugate term. However, transformation $\phi_n \to \phi_n \mathrm{e}^{\mathrm{i}\theta}$is valid in the parametrically driven DNLS (1) for $\theta = \pi + 2k\pi\theta$ and $\theta = -\pi/2 + 2k\pi\theta$ with $k \in \mathbb{Z}$, which lead, respectively, to the reflection symmetry under transformations

$$\phi_n \to -\phi_n \quad (2)$$

and

$$\phi_n \to -\mathrm{i}\phi_n, \quad \gamma \to -\gamma. \quad (3)$$

Consequently, transformation (3) allows us to only consider the driving constant $\gamma > 0$.

In this paper, we are particularly interested in steady-state localized solutions of Eq. (1) having the form $\phi_n(t) = u_n$where u_n is complex-valued and time-independent. In this case, u_n satisfies the stationary equation

$$-\varepsilon\Delta_2 u_n + \Lambda u_n + \gamma\bar{u}_n - |u_n|^2 u_n = 0, \tag{4}$$

under localization condition

$$u_n \to 0 \text{ as } n \to \pm\infty, \tag{5}$$

which corresponds to discrete bright soliton solutions. One should notice that Λ in Eq. (1) can be scaled out to 1 by transformation

$$u_n \to u_n\sqrt{\Lambda}, \quad \varepsilon \to \varepsilon\Lambda, \quad \gamma \to \gamma\Lambda. \tag{6}$$

Thus, in what follows, we set $\Lambda = 1$.

In the absence of parametric driving, i.e., when $\gamma = 0$, it was shown, e.g., in [17,18] that all localized solutions of Eq. (4) satisfying the conditions (5) are real-valued. In the presence of parametric driving, i.e., when $\gamma > 0$, it has been shown in [19] that there are only two possibilities for a localized solution of the stationary PDNLS (4), i.e., either real (provided $\Lambda > -\gamma$) or purely imaginary (provided $\Lambda > \gamma$), where the latter is always unstable. One can check that the purely imaginary solution can always be obtained from the real one through transformation (3), so the study in this paper will be devoted for real-valued solutions only.

3. The Variational Approximation

In this section we derive a variational approximation (VA) for the steady-state localized solutions of the stationary equation (4). Recall that the aforementioned equation can be represented in the variational form

$$\frac{\delta L}{\delta \bar{u}_n} \equiv \frac{\partial L}{\partial \bar{u}_n} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\bar{u}}_n} = 0, \tag{7}$$

where L is the Lagrangian given by

$$L = \sum_{n=-\infty}^{\infty} \varepsilon(\bar{u}_n u_{n+1} + u_n \bar{u}_{n+1}) - (2\varepsilon + 1)|u_n|^2 - \frac{\gamma}{2}(\bar{u}_n^2 + u_n^2) + \frac{1}{2}|u_n|^4. \tag{8}$$

As we consider real-valued solutions, we can simplify the Lagrangian (8) to be

$$L = \sum_{n=-\infty}^{\infty} \varepsilon(u_n u_{n+1} + u_n u_{n+1}) - (2\varepsilon + 1)u_n^2 - \gamma u_n^2 + \frac{1}{2}u_n^4. \tag{9}$$

Note that in the uncoupled limit $\varepsilon = 0$, Eq. (4) permits the exact solutions $u_n = u_n^{(0)}$ in which each $u_n^{(0)}$ takes one of the three values 0, $\pm\sqrt{\gamma + 1}$. For configuration of a twisted soliton, its mode structure is of the form

$$u_n^{(0)} = \begin{cases} \sqrt{\gamma + 1}, & n = 0, \\ -\sqrt{\gamma + 1}, & n = 1, \\ 0, & \text{otherwise.} \end{cases} \tag{10}$$

Recall that due to the reflection symmetry (2), we neglect the negative quantity of the excited sites.

Therefore, to find approximate solutions for twisted solitons which can take into account the case of the AC limit, we propose an ansatz in the form

$$u_n = \begin{cases} B,\ n = 0, \\ -B,\ n = 1, \\ Ae^{-a\left(-n+\frac{1}{2}\right)},\ n < 0, \\ -Ae^{-a\left(n-\frac{1}{2}\right)},\ n > 1, \end{cases} \tag{11}$$

where $B > 0$, $A > 0$, and a are real variational parameters.

After inserting the ansatz (11) into the Lagrangian (9) and then performing the sums, we end up with the effective Lagrangian

$$L_{\text{eff}} = \frac{\mathrm{e}^{-2a}A^4}{\mathrm{e}^{4a}-1} - 2A^2\frac{(\gamma+2\varepsilon+1)e^{a} - \varepsilon\mathrm{e}^{-2a} + (\gamma+2\varepsilon+1)\mathrm{e}^{-a} - \mathrm{e}^{2a}\varepsilon - 2\varepsilon}{\mathrm{e}^{4a}-1} - 2\varepsilon AB\mathrm{e}^{\frac{3a}{2}} - 2\varepsilon B^2$$

$$-2(1+2\varepsilon)B^2 - 2\gamma B^2 + B^4. \tag{12}$$

From the variational principle, the effective Lagrangian L_{eff} achieves critical values at the Euler-Lagrange equations

$$\frac{\partial L_{\text{eff}}}{\partial B} = \frac{\partial L_{\text{eff}}}{\partial A} = \frac{\partial L_{\text{eff}}}{\partial a} = 0. \tag{13}$$

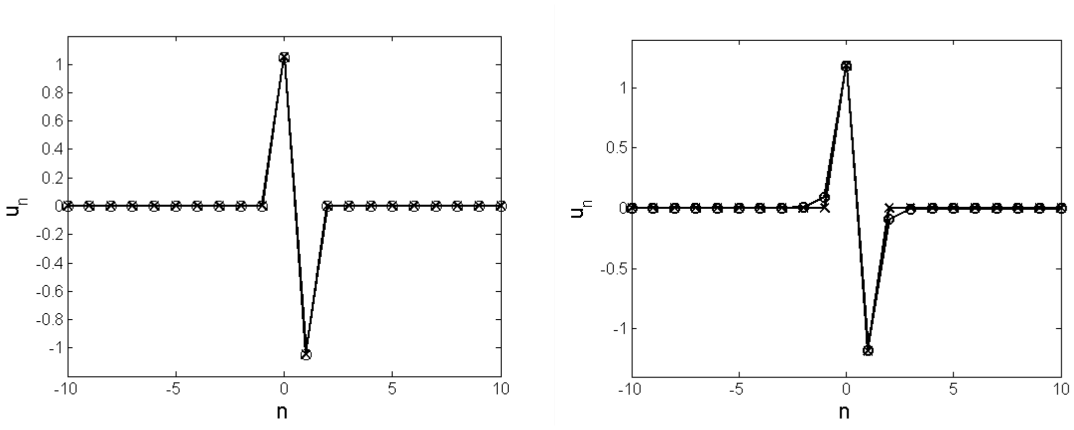
Thus, the next step is to substitute Eq. (11) into Eq. (12) from which we obtain a system of equations (since its complicated expression, we do not write the system here). Due to its complicity, for given parameters ε and γ, we solve the system numerically to obtain solutions for variational parameters A_0, A, and a. The resulting solutions then give an approximation for the soliton described by the ansatz (11).

4. The VA results and comparisons with numerics

In this section, we compare the results of variational approximations with the corresponding numerical findings. The numerical solutions of the stationary equation (7) are calculated using a Newton-Raphson method with the VA solutions being initial guess.

As an illustrative example, we present in Figure 1 the comparison between two soliton profiles obtained from the numerical results and VA for parametric driving $\gamma = 0.1$ and two values $\varepsilon = 0, 0.3$. After solving the system of equations, which is obtained from at the Euler-Lagrange equation (13), for the aforementioned parameter values, we obtained $(B, A, a) \approx (1.0488, 0,0)$ for the uncoupled limit $\varepsilon = 0$ [Figure 1(a)] and $(B, A, a) \approx (1.1832, 0, 0)$ for $\varepsilon = 0.3$ [Figure 1(b)]. We observe that our approximations are generally in good agreement with the numerics.

To further confirm the validity of our VA, we plot in Figure 2 the norm of numerical solution of the twisted soliton and the corresponding analytical approximation as functions of ε for two values $\gamma = 0.1, 0.5$.. As seen in the figure, the approximation given by VA for $\gamma = 0.5$ is better than the one with $\gamma = 0.1$.



(a) $(\varepsilon,\gamma) = (0,0.1)$ (b) $(\varepsilon,\gamma) = (0.3,0.1)$

Figure 1. Comparison between numerical (circle markers) and variational (cross markers) solutions of the twisted soliton with parameter values as indicated in the caption of each panel.

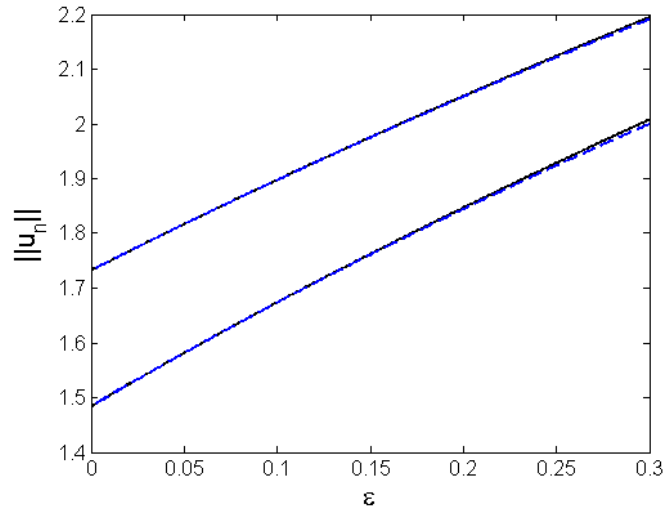


Figure 2. The norm of the twisted solitons calculated by numerics (solid lines) and VA (dashed lines) in varied ε. The upper and lower curves correspond, respectively, to $\gamma = 0.5$ and $\gamma = 0.1$.

Based on our observation to Figure 2, we can conclude that our VA is excellent in the vicinity of the AC limit. It means that the developed VA can be used to approximate the stationary twisted solitons for small coupling constant ε. Furthermore, from the figure we also observe that our VA is getting more accurate as parametric driving γ increases. Indeed, these results must be justified rigorously by, e.g., the validity of the VA developed in [10].

5. Conclusion

In this paper, we have developed a variational formulation for approximation of the stationary twisted solitons governed by the parametrically driven discrete nonlinear Schrödinger equation. An ansatz that can deal with the anticontinuum limit case has been proposed. The comparison between the analytical results and the corresponding numerical findings showed that our theoretical predictions are very good for small coupling constant ε where the accuracy is getting better for larger parametric driving γ. However, one should notice that the proposed ansatz (11) can not capture the smooth growth of the twisted solitons as ε increases. Therefore, the better ansatz for this case should be developed and the heuristic validity must be performed rigorously These interesting problems can be proposed as future study.

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