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Linear Quadratic Optimization for Positive LTI System

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Abstract. Nowaday the linear quadratic optimization subject to positive linear time invariant (LTI) system constitute an interesting study considering it can become a mathematical model of variety of real problem whose variables have to nonnegative and trajectories generated by these variables must be nonnegative. In this paper we propose a method to generate an optimal control of linear quadratic optimization subject to positive linear time invariant (LTI) system. A sufficient condition that guarantee the existence of such optimal control is discussed.

INTRODUCTION

Given the following linear time invariant (LTI) system:

$$\begin{aligned} \dot{\mathbf{x}} &= A\mathbf{x} + B\mathbf{u} + E\boldsymbol{\omega}, \quad \mathbf{x}(0) = \mathbf{x}_0 \\ \mathbf{y} &= C\mathbf{x} + F\boldsymbol{\omega}, \end{aligned} \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^n$ is the state vector, $\mathbf{y} \in \mathbb{R}^r$ is the output vector, $\mathbf{u} \in \mathbb{R}^m$ is the input vector, $\boldsymbol{\omega} \in \mathbb{R}^q$ is disturbance vector, $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{r\times n}, E \in \mathbb{R}^{n\times q}$ and $F \in \mathbb{R}^{r\times q}$. The system (1) is called positive if for each nonnegative initial state, nonnegative input and nonnegative disturbance imply the state and output are nonnegative for every nonnegative time, i.e. $\mathbf{x} \in \mathbb{R}_+^n$ and $\mathbf{y} \in \mathbb{R}_+^r$ for every $t \in \mathbb{R}_+$. It is well known that system (1) is positive if and only if A is a Metzler matrix, $B \in \mathbb{R}_+^{n\times m}, C \in \mathbb{R}_+^{r\times n}, E \in \mathbb{R}_+^{n\times q}$ and $F \in \mathbb{R}_+^{r\times q}$ [1]. The characterization of positive systems attracts interest because this kind of system appears in the modelling of many processes in various field, e.g. in biology, chemistry and economics [4, 6]. In these models state variables represent population, measure, mass, etc., and therefore, they are nonnegative. Many aspects of positive linear systems have been considered by different authors. A complete introduction to positive linear systems can be found in Farina and Rinaldi [3]. Stability of the positive linear system has been studied in Leenheer and Aeyels [5]. It turns out that system (1) is asymptotic stable if and only if A is Hurwitz, namely, $Re(\lambda)$ for all $\lambda \in \sigma(A)$.

Recently, Wu, et al. [9] discussed the linear quadratic optimization subject to LTI system with disturbance without positiveness constraint of the system (1). The problem is to determine a control $\mathbf{u} \in \mathbb{R}^m$ that satisfy (1) and to minimize the following quadratic performance index:

$$J(\mathbf{u}) = \int_0^\infty (\gamma^2\, \mathbf{x}^{\mathrm{T}} S \mathbf{x} + \mathbf{u}^{\mathrm{T}}\mathbf{u})\, dt,$$

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where $\gamma > 0$ is a parameter and $S \in \mathbb{R}^{n\times n}$ is a symmetric positive semidefinite. Moreover, Roszak and Davidson [7], Beauthier and Winkin [2] also discussed other aspect of the linear quadratic optimization subject to the system (1) but considering positiveness constraint.

In practice, output does not always behave as desired. This occurs because of there are some disruption in the system. Therefore, the objective function can be modified into another form. In this short note, we extend the results from Roszak and Davidson [7] by eliminating the assumption asymptotic stability of the system (1).

PROBLEM FORMULATION

Given the system (1) where state $\mathbf{x}$ is unmeasurable and the initial condition $\mathbf{x}(0) \in \mathbb{R}_+^n$. Assume that the system (1) is positive, $\text{rank}(A) = n,\ r = m$, i.e. the number of inputs is equal to the number of outputs, and $\mathbf{y_{des}} \in \mathbb{R}_+^r$ is a constant desired output. Find an optimal LTI controller $\mathbf{u}(t) = K\mathbf{x}(t) \in \mathbb{R}^m$ for some $k \in \mathbb{R}^{m\times n}$ that minimize the following objective function:

$$\Im(\mathbf{u}) = \int_0^\infty (\gamma^2\, \boldsymbol{\epsilon}^{\mathrm{T}} Q \boldsymbol{\epsilon} + \dot{\mathbf{u}}^{\mathrm{T}} \dot{\mathbf{u}})\, dt, \tag{2}$$

where $\gamma > 0$ is a parameter and $Q \in \mathbb{R}^{r\times r}$ is a symmetric positive definite, and closed system

$$\begin{aligned} \dot{\mathbf{x}} &= (A + BK)\mathbf{x} + E\boldsymbol{\omega},\ \ \mathbf{x}(0) = \mathbf{x}_0, \\ \mathbf{y} &= C\mathbf{x} + F\boldsymbol{\omega}, \end{aligned} \tag{3}$$

is

1. asymptotic stable;
2. the state $\mathbf{x} \in \mathbb{R}_+^n$ and the output $\mathbf{y} \in \mathbb{R}_+^r$ for every $t \in \mathbb{R}_+$;
3. error $\boldsymbol{\epsilon} = (\mathbf{y} - \mathbf{y_{des}}) \to \mathbf{0}$ as $t \to \infty$.

Note that in this case we do not need assume that the system (1) is asymptotic stable; compare to the assumption in Roszak and Davidson [7].

MAIN RESULT

In the following we present the process to construct the optimal control $\mathbf{u}_{\text{opt}}(t)$. Assume that $\text{rank}(-CA^{-1}B) = r$ and define $K_0 = (-CA^{-1}B)^{-1}$ and $K_1 = -K_0(F - CA^{-1}E)$. It is easy to show that for any desire output $\mathbf{y_{des}}$ and disturbance $\boldsymbol{\omega}$, the steady state control $\mathbf{u_s}$ satisfies

$$\mathbf{u_s} = K_0\mathbf{y_{des}} + K_1\boldsymbol{\omega},$$

if the optimal control $\mathbf{u}_{\text{opt}}(t)$ exists [7].

Define a new variable $\boldsymbol{\varphi}$ such that $\dot{\boldsymbol{\varphi}} = \gamma\boldsymbol{\epsilon}$, the system (1) can be written as follows:

$$\begin{pmatrix} \dot{\mathbf{x}} \\ \boldsymbol{\epsilon} \end{pmatrix} = \begin{pmatrix} A & O \\ C & O \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \boldsymbol{\varphi} \end{pmatrix} + \begin{pmatrix} B \\ O \end{pmatrix} \mathbf{u} + \begin{pmatrix} E \\ F \end{pmatrix} \boldsymbol{\omega} - \begin{pmatrix} O \\ I_r \end{pmatrix} \mathbf{y}_{\text{des}}. \tag{4}$$

Since the disturbance $\boldsymbol{\omega}$ and the desire output $\mathbf{y}_{\text{des}}$ are constant, (1) can be written as

$$\begin{pmatrix} \ddot{\mathbf{x}} \\ \dot{\boldsymbol{\epsilon}} \end{pmatrix} = \begin{pmatrix} A & O \\ C & O \end{pmatrix} \begin{pmatrix} \dot{\mathbf{x}} \\ \gamma\boldsymbol{\epsilon} \end{pmatrix} + \begin{pmatrix} B \\ O \end{pmatrix} \dot{\mathbf{u}}. \tag{5}$$

Define $Q = K_0^{\mathrm{T}} K_0$. Likewise the problem under consideration becomes to find the control $\mathbf{u}$ that satisfy (5) and to minimize the objective function (2) and simultaneously to satisfy the condition 1-3. In other form this optimization problem can be written as follows:

$$\min_{\mathbf{v}} \quad \int_0^{\infty} (\boldsymbol{\phi}^{\mathrm{T}} \bar{Q} \boldsymbol{\phi} + \mathbf{v}^{\mathrm{T}} \mathbf{v})\, dt \tag{6}$$

$$\text{s.t. } \dot{\boldsymbol{\phi}} = \bar{A}\, \boldsymbol{\phi} + \bar{B}\mathbf{v},$$

where

$$\mathbf{v} = \dot{\mathbf{u}},\ \boldsymbol{\phi} = \begin{pmatrix} \dot{\mathbf{x}} \\ \boldsymbol{\epsilon} \end{pmatrix}, \bar{A} = \begin{pmatrix} A & O \\ C & O \end{pmatrix}, \bar{B} = \begin{pmatrix} B \\ O \end{pmatrix} \text{ and } \bar{Q} = \begin{pmatrix} O & O \\ O & \gamma^2 Q \end{pmatrix}. \tag{7}$$

Base on the LQR theory [8], if the pair $(\bar{A}, \bar{B})$ is stabilizable and the pair $(\bar{Q}, \bar{A})$ is detectable then the optimal control for the problem (6) is

$$\mathbf{v}_{\mathrm{opt}} = -(P\bar{B})^{\mathrm{T}} \boldsymbol{\phi}_{\mathrm{opt}}, \tag{8}$$

where P is a symmetric positive definite matrix that constitute a solution of the following algebraic Riccati equation:

$$\bar{A}^{\mathrm{T}} P + P\bar{A} + \bar{Q} - (P\bar{B})(P\bar{B})^{\mathrm{T}} = O, \tag{9}$$

and $\boldsymbol{\phi}_{\mathrm{opt}}$ is the corresponding optimal state that satisfy the following differential equation:

$$\dot{\boldsymbol{\phi}} = (\bar{A} - \bar{B}^{\mathrm{T}} P)\boldsymbol{\phi},$$

where $\boldsymbol{\phi}_{\mathrm{opt}} \to \mathbf{0}$ as $t \to \infty$. This implies $\boldsymbol{\epsilon} \to \mathbf{0}$ and $\mathbf{x}_{\mathrm{opt}} \to \mathbf{0}$ that proving the condition 1 and 3.

On applying (7) we have

$$\dot{\mathbf{u}}_{\mathrm{opt}} = (K_1 \quad K_2) \begin{pmatrix} \dot{\mathbf{x}}_{\mathrm{opt}} \\ \boldsymbol{\epsilon} \end{pmatrix}, \tag{10}$$

where $(K_1 \quad K_2) = -(P\bar{B})^{\mathrm{T}}, K_1 \in \mathbb{R}^{m \times n}$ and $K_2 \in \mathbb{R}^{m \times r}$ such that

$$\mathbf{u}_{\mathrm{opt}} = (K_1 \quad K_2) \begin{pmatrix} \mathbf{x}_{\mathrm{opt}} \\ \mathbf{0} \end{pmatrix}.$$

So, we can choose $K = K_1$. Furthermore, using Corollary 3.7 in Beauthier and Winkin [2], the closed loop (1) is positive if the matrix $A + BK_1$ is Metzler. This prove the condition 2.

CONCLUSION

We have already show that desire optimal control for the problem under consideration is given by

$$\mathbf{u}_{\mathrm{opt}} = K_1 \mathbf{x}_{\mathrm{opt}}.$$

This control makes the closed loop (3) is asymptotic stable, positive and error $\boldsymbol{\epsilon} = (\mathbf{y} - \mathbf{y}_{\mathrm{des}}) \to \mathbf{0}$ as $t \to \infty$. Some open problems of this note are:

1. under what condition the pair $(\bar{A}, \bar{B})$ is stabilizable and the pair $(\bar{Q}, \bar{A})$ is detectable.
2. under what condition the matrix $A + BK_1$ is Metzler matrix.

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